Knowledge-Based Systems 28 (2012) 13–18

Contents lists available at SciVerse ScienceDirect

Knowledge-Based Systems

journal homepage: www.elsevier.com/locate/knosys

On the axiomatization of some classes of discrete universal integrals

Erich Peter Klement [a,*], Radko Mesiar [b,c]

[a] Department of Knowledge-Based Mathematical Systems, Johannes Kepler University, Linz, Austria
[b] Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology, Bratislava, Slovakia
[c] Institute of Theory of Information and Automation, Czech Academy of Sciences, Prague, Czech Republic

ARTICLE INFO

Article history:
Received 20 June 2011
Received in revised form 25 October 2011
Accepted 26 October 2011
Available online 7 November 2011

Keywords:
Comonotone modularity
Copula
Universal integral
Choquet integral
Sugeno integral

ABSTRACT

Following the ideas of the axiomatic characterization of the Choquet integral due to [D. Schmeidler, Integral representation without additivity, Proc. Amer. Math. Soc. 97 (1986) 255–261] and of the Sugeno integral given in [J.-L. Marichal, An axiomatic approach of the discrete Sugeno integral as a tool to aggregate interacting criteria in a qualitative framework, IEEE Trans. Fuzzy Syst. 9 (2001) 164–172], we provide a general axiomatization of some classes of discrete universal integrals, including the case of discrete copula-based universal integrals (as usual, the product copula corresponds just to the Choquet integral, and the minimum to the Sugeno integral).

1. Introduction

In this contribution, we restrict ourselves to a fixed finite space $X = \{1, \ldots, n\}$, and we will deal with functions from X to $[0,1]$ which we identify with n-dimensional vectors $\mathbf{x} = (x_1, \ldots, x_n)$. From the application point of view, we can look at $\mathbf{x}$ as a score vector of some alternative characterized by n criteria. To be able to decide which of the alternatives described by the score vectors $\mathbf{x}$ and $\mathbf{y}$, respectively, is to be preferred, a typical approach is to evaluate both $\mathbf{x}$ and $\mathbf{y}$ by means of some utility function U.

The utility function U is often constructed from a boolean utility function B acting on $\mathbf{x} \in \{0,1\}^n$. However, each such boolean utility function B can be seen as a capacity $m : 2^X \to [0,1]$, $m(E) = B(\mathbf{1}_E)$. Typical extension approaches are related to integration, i.e., $U(\mathbf{x}) = \mathbf{I}(m,\mathbf{x})$, where $\mathbf{I}(m,\cdot)$ is some integral on X with respect to the capacity m.

Another approach is based on some axiomatization (and boolean utility function B). It is well-known that the additivity of the utility function $U : [0,1]^n \to [0,1]$ is related to the application of Lebesgue integral, $U(\mathbf{x}) = \int \mathbf{x}\, dm$, and then also the capacity m should be additive. Putting $m(\{i\}) = w_i$, we obtain the well-known weighted arithmetic mean, $U(\mathbf{x}) = \sum_{i=1}^{n} w_i \cdot x_i$.

Our contribution recalls some classes of universal integrals (including, among others, the Choquet, the Sugeno and the Lebesgue integral) and provides corresponding axiomatizations.

Because of the link to utility functions, we restrict ourselves to (normed) capacities and to input values from $[0,1]$, although many integrals mentioned here (including the Choquet and Sugeno integral) can be considered in a more general (unbounded) framework [14].

However, we do not consider any further restriction concerning the underlying capacity, such as additivity or pseudo-additivity, and thus we will not deal with integrals based on such special capacities (compare, e.g., [20,22,27]).

The paper is organized as follows. In the following section, the Choquet and the Sugeno integral as well as their axiomatizations are summarized. In Section 3, we recall (discrete) copula-based integrals and some other classes of discrete universal integrals, including some examples. In Section 4, the axiomatization of these discrete universal integrals is given. As a special case, symmetric discrete copula-based universal integrals (generalizing OWA operators) are discussed.

2. Choquet and Sugeno integrals, and their axiomatization

Though all integrals discussed in this paper can be defined on an arbitrary measurable space, in this paper we consider (as already mentioned) the finite space $X = \{1, \ldots, n\}$ only, equipped with the σ-algebra $2^X = \{E | E \subseteq X\}$.

Definition 2.1. A *capacity* on X is a set function $m : 2^X \to [0,1]$ which is non-decreasing, i.e., we have $m(E) \leqslant m(F)$ whenever $E \subseteq F \subseteq X$, and satisfies the boundary conditions $m(\emptyset) = 0$ and $m(X) = 1$.

* Corresponding author.
E-mail addresses: ep.klement@jku.at (E.P. Klement), mesiar@math.sk (R. Mesiar).

0950-7051/$ - see front matter © 2011 Elsevier B.V. All rights reserved.
doi:10.1016/j.knosys.2011.10.015

Then the *Choquet integral* [3] of $\mathbf{x}$ with respect to a capacity $m : 2^X \to [0,1]$ is defined by

$$\mathbf{Ch}(m,\mathbf{x}) = \int_0^1 m(\{i|x_i \geqslant t\})\, dt$$
$$= \sum_{i=1}^{n} x_{\pi_i} \cdot (m(\{\pi_i,\ldots,\pi_n\}) - m(\{\pi_{i+1},\ldots,\pi_n\})), \qquad (1)$$

for some permutation $(\pi_1,\pi_2,\ldots,\pi_n)$ of $\{1,\ldots,n\}$ satisfying $x_{\pi_1} \leqslant x_{\pi_2} \leqslant \cdots \leqslant x_{\pi_n}$, where the set $\{\pi_{n+1},\pi_n\}$ occurring in the last summand is defined to be the empty set $\emptyset$.

Obviously, we have $m(E) = \mathbf{Ch}(m,\mathbf{1}_E)$ for each $E \subseteq X$. Observe that if m is additive (i.e., m is a discrete probability measure) then the Choquet integral coincides with the Lebesgue integral, i.e., $\mathbf{Ch}(m,\mathbf{x}) = \int \mathbf{x}\, dm$.

Similarly, the *Sugeno integral* [26] of $\mathbf{x}$ with respect to a capacity $m : 2^X \to [0,1]$ is given by

$$\mathbf{Su}(m,\mathbf{x}) = \bigvee_{t=0}^{1}(t \wedge m(\{i|x_i \geqslant t\})) = \bigvee_{i=1}^{n}(x_{\pi_i} \wedge m(\{\pi_i,\ldots,\pi_n\})). \qquad (2)$$

Note that we use the symbols $\wedge$ and $\vee$ in the sense $x \wedge y = \min(x,y)$ and $x \vee y = \max(x,y)$. Clearly, also for the Sugeno integral we have $m(E) = \mathbf{Su}(\mathbf{1}_E)$ for all $E \subseteq X$.

In what follows, the comonotonicity of score vectors plays a crucial role.

Definition 2.2 [23]. Let $\mathbf{x}, \mathbf{y} \in [0,1]^n$. Then $\mathbf{x}$ and $\mathbf{y}$ are said to be *comonotone* if, for all i, $j \in \{1,2,\ldots,n\}$, we have $(x_i - x_j)\cdot(y_i - y_j) \geqslant 0$.

In other words, for comonotone $\mathbf{x}, \mathbf{y} \in [0,1]^n$ it is impossible to have $x_i > x_j$ and $y_i < y_j$. In [23] an axiomatic characterization of the Choquet integral as a comonotone aggregation function [4,8] was given.

Definition 2.3

(i) An (n-dimensional) *aggregation function* is a function $A : [0,1]^n \to [0,1]$ which is non-decreasing in each component and satisfies the boundary conditions $A(0,\ldots,0) = 0$ and $A(1,\ldots,1) = 1$.
(ii) An aggregation function $A : [0,1]^n \to [0,1]$ is said to be *comonotone additive* if, for all $\mathbf{x}$, $\mathbf{y} \in [0,1]^n$ which are comonotone and satisfy $\mathbf{x} + \mathbf{y} \in [0,1]^n$, we have
$$U(\mathbf{x}+\mathbf{y}) = U(\mathbf{x}) + U(\mathbf{y}).$$

Observe that the comonotone additivity of an aggregation function U implies its positive homogeneity, i.e., $U(c\cdot\mathbf{x}) = c\cdot U(\mathbf{x})$ for all $c \geqslant 0$ and $\mathbf{x} \in [0,1]^n$ with $c\cdot\mathbf{x} \in [0,1]^n$.

Proposition 2.4 [23]. *Let $U : [0,1]^n \to [0,1]$ be an n-ary aggregation function. Then the following are equivalent:*

(i) *There is a capacity $m : 2^X \to [0,1]$ such that $U(\cdot) = \mathbf{Ch}(m,\cdot)$.*
(ii) *U is comonotone additive.*

In the case of Sugeno integral, its axiomatization was given in [16].

Proposition 2.5 [16]. *Let $U : [0,1]^n \to [0,1]$ be an n-ary aggregation function. Then the following are equivalent:*

(i) *There is a capacity $m : 2^X \to [0,1]$ such that $U(\cdot) = \mathbf{Su}(m,\cdot)$.*
(ii) *U is $\wedge$-homogeneous and comonotone maxitive, i.e., for each $c \in [0,1]$, the constant score vector $\mathbf{c} = (c,\ldots,c)$ and all comonotone $\mathbf{x}$, $\mathbf{y} \in [0,1]^n$ we have*
$$U(\mathbf{c} \wedge \mathbf{x}) = c \wedge U(\mathbf{x}),$$
$$U(\mathbf{x} \vee \mathbf{y}) = U(\mathbf{x}) \vee U(\mathbf{y}).$$

Observe that the comonotone maxitivity of an aggregation function U does not imply its $\wedge$-homogeneity. Note that there are some alternative axiomatic approaches to the Sugeno integral (compare [1,16]).

3. Some classes of discrete universal integrals

We briefly recall some classes of discrete universal integrals which will be characterized in an axiomatic way in Section 4. For functions with values in the nonnegative real numbers, the concept of a *universal integral* which can be defined on arbitrary (not necessarily finite) measurable spaces and for arbitrary capacities, was introduced axiomatically and investigated in [14]. It is based on a special type of binary aggregation function, the so-called semicopula [5].

Definition 3.1. A *semicopula* is two-dimensional aggregation function $\odot : [0,1]^2 \to [0,1]$ with neutral element 1.

Given a semicopula $\odot[0,1]^2 \to [0,1]$ and a capacity $m: \to [0,1]$ we will require that each discrete universal integral acts on $[0,1]^n$ as a special aggregation function.

Definition 3.2. Let $\odot : [0,1]^2 \to [0,1]$ be a semicopula and let $m : 2^X \to [0,1]$ be a capacity on X. A *discrete universal integral* (*based on* $\odot$) is an aggregation function $\mathbf{I}_{\odot,m} : [0,1]^n \to [0,1]$ such that

(i) for all $c \in [0,1]$ and all $E \subseteq X$ we have $\mathbf{I}_{\odot,m}(c\cdot\mathbf{1}_E) = c \odot m(E)$;
(ii) for all $\mathbf{x}$, $\mathbf{y} \in [0,1]^n$ with $m(\{i \in X|x_i \geqslant t\}) = m(\{j \in X|y_j \geqslant t\})$ for all $t \in [0,1]$ we have $\mathbf{I}_{\odot,m}(\mathbf{x}) = \mathbf{I}_{\odot,m}(\mathbf{y})$.

Note that each discrete universal integral as given in Definition 3.2 is an idempotent aggregation function because of

$$\mathbf{I}_{\odot,m}(c,c,\ldots,c) = \mathbf{I}_{\odot,m}(c\cdot\mathbf{1}_X) = c \odot m(X) = c \odot 1 = c.$$

Observe that if a capacity m assumes values in $\{0,1\}$ only then all discrete universal integrals are independent of the semicopula $\odot$, and they correspond to lattice polynomials (compare [15]). Moreover, the class of discrete universal integrals is convex, i.e., for each monotone measure m, for all discrete universal integrals $\mathbf{I}^{(1)}_{\odot_1,m}$ and $\mathbf{I}^{(2)}_{\odot_2,m}$ based on the semicopulas $\odot_1$ and $\odot_2$, respectively, and for each $\lambda \in [0,1]$, also

$$\mathbf{I}_{\odot,m} = \lambda\cdot\mathbf{I}^{(1)}_{\odot_1,m} + (1-\lambda)\cdot\mathbf{I}^{(2)}_{\odot_2,m}$$

is a discrete universal integral based on the semicopula $\odot = \lambda\cdot\odot_1 + (1-\lambda)\cdot\odot_2$.

3.1. Discrete copula-based universal integrals

Universal integrals (acting on the interval $[0,\infty]$) were introduced and discussed in [14]. A special kind of universal integrals on the scale $[0,1]$ is based on copulas [21,25], compare also [13].

Definition 3.3. A (*binary*) *copula* $C : [0,1]^2 \to [0,1]$ is a semicopula which is supermodular, i.e., for all $\mathbf{x}, \mathbf{y} \in [0,1]^2$

$$C(\mathbf{x} \vee \mathbf{y}) + C(\mathbf{x} \wedge \mathbf{y}) \geqslant C(\mathbf{x}) + C(\mathbf{y}). \qquad (3)$$

We are not going into details about universal integrals and copulas here, we only recall the following important result (see Remark 5.3, 2 in [14]):

Proposition 3.4 [14]. *Let $C : [0,1]^2 \to [0,1]$ be a copula and $m : 2^X \to [0,1]$ a capacity, and define $\mathbf{K}_C(m,\cdot) : [0,1]^n \to [0,1]$ by*

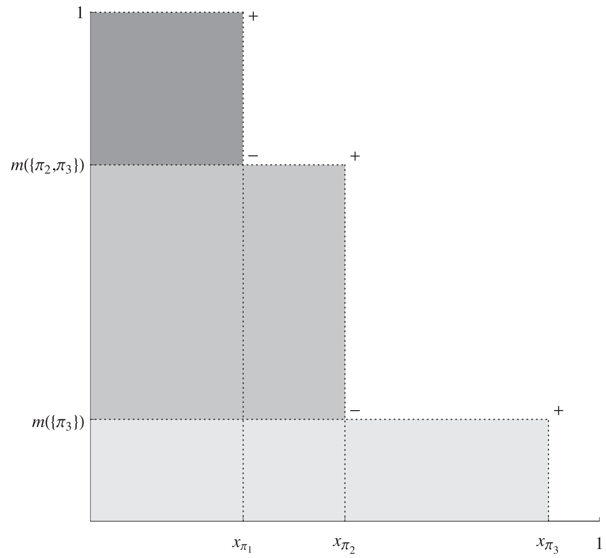

Fig. 1. Copula-based universal integral K_C.

$$\mathbf{K}_C(m,\mathbf{x}) = \sum_{i=1}^{n} C(x_{\pi_i}, m(\{\pi_i,\ldots,\pi_n\})) - C(x_{\pi_{i-1}}, m(\{\pi_i,\ldots,\pi_n\})), \quad (4)$$

putting $x_{(\pi_0)} = 0$*, by convention. Then* $\mathbf{K}_C$ *is a discrete universal integral.*

Fig. 1 indicates how a copula-based universal integral is computed in the case $n = 3$:

$$\mathbf{K}_C(m,\mathbf{x}) = C(x_{\pi_1}, 1) - C(x_{\pi_1}, m(\{\pi_2,\pi_3\})) + C(x_{\pi_2}, m(\{\pi_2,\pi_3\})) - C(x_{\pi_2}, m(\{\pi_3\})) + C(x_{\pi_3}, m(\{\pi_3\})).$$

It is not difficult to show that also the following formula – which is equivalent to (4) – holds:

$$\mathbf{K}_C(m,\mathbf{x}) = \sum_{i=1}^{n} \left(C(x_{\pi_i}, m(\{\pi_i,\ldots,\pi_n\})) - C(x_{\pi_i}, m(\{\pi_{i+1},\ldots,\pi_n\}))\right). \quad (5)$$

However, then for the product copula Π, $\Pi(x,y) = x \cdot y$, the formula (5) turns into (1),

$$\mathbf{K}_\Pi(m,\mathbf{x}) = \sum_{i=1}^{n} x_{\pi_i} \cdot (m(\{\pi_i,\ldots,\pi_n\}) - m(\{\pi_{i+1},\ldots,\pi_n\})),$$

i.e., $\mathbf{K}_\Pi$ coincides with the Choquet integral **Ch**. Similarly, for the greatest copula M given by $M(x,y) = x \wedge y$, formula (5) turns into (2), and thus $\mathbf{K}_M$ is just the Sugeno integral **Su**.

Observe that the class of copulas is convex and, therefore, for each $\lambda \in [0,1]$, the function $C_\lambda = \lambda \cdot \Pi + (1 - \lambda) \cdot M$ is a copula. An immediate consequence is that $\mathbf{K}_{C_\lambda} = \lambda \cdot \mathbf{K}_\Pi + (1-\lambda) \cdot \mathbf{K}_M = \lambda \cdot \mathbf{Ch} + (1-\lambda) \cdot \mathbf{Su}$, i.e., a convex combination of the Choquet and the Sugeno integral.

Moreover, for each copula C, the function $\widehat{C} : [0,1]^2 \to [0,1]$ given by $\widehat{C}(x,y) = x + y - 1 + C(1-x, 1-y)$ is also a copula ($\widehat{C}$ is called a *survival copula* [21]). Then we get (see [13])

$$\mathbf{K}_{\widehat{C}}(m,\mathbf{x}) = 1 - \mathbf{K}_C(m^d, 1 - \mathbf{x}),$$

where $m^d : 2^X \to [0,1]$ is the *dual capacity of* m given by $m^d(E) = 1 - m(X \backslash E)$ (in the language of aggregation functions, the integral $\mathbf{K}_{\widehat{C}}$ with respect to a capacity m is dual to the integral $\mathbf{K}_C$ with respect to the dual capacity m^d). Because of $\hat{\Pi} = \Pi$ and $\widehat{M} = M$ we have

$$\mathbf{Ch}(m,\mathbf{x}) = 1 - \mathbf{Ch}(m^d, 1 - \mathbf{x}) \quad \text{and} \quad \mathbf{Su}(m,\mathbf{x}) = 1 - \mathbf{Su}(m^d, 1 - \mathbf{x}).$$

Example 3.5. Consider the smallest copula $W : [0,1]^2 \to [0,1]$ given by $W(x,y) = (x + y - 1) \vee 0$, $X = \{1,2\}$, and let $m : 2^X \to [0,1]$ be a capacity satisfying, for some $\alpha \in [0,1]$, $m(\{1\}) = m(\{2\}) = \alpha$. Then $\mathbf{K}_W(m,\cdot)$ is a universal integral with neutral element $1 - \alpha$, and it is given by $\mathbf{K}_W(m,\mathbf{x}) = \text{med}(x_1, x_2, x_1 + x_2 + \alpha - 1)$ (as usual, med (x,y,z) denotes the *median* of x, y and z). Note that $\mathbf{K}_W(m,\mathbf{x}) = x_1 \wedge x_2$ if $\alpha = 0$, and $\mathbf{K}_W(m,\mathbf{x}) = x_1 \vee x_2$ if $\alpha = 1$.

3.2. Discrete Benvenuti integrals

A special type of universal integral acting on the scale $[0,1]$ is linked to the Benvenuti integral which is described in detail in [1]. For a fixed interval $[0,a]$ with $a \in [1,\infty]$, let $\oplus : [0,a]^2 \to [0,a]$ be a *pseudo-addition on* $[0,a]$, i.e., a binary operation which is continuous, non-decreasing, associative and has neutral element 0. Monotonicity and neutral element 0 imply that $a \oplus a = a$, therefore Proposition 2.41 in [12] tells us that $([0,a],\oplus)$ is a so-called *I*-semigroup [19], i.e., the operation $\oplus$ is not only continuous, non-decreasing, associative and has neutral element 0, but also has annihilator a. As a consequence, $\oplus$ is also symmetric (this follows, e.g., from [12, Theorem 2.43]).

Define a $\oplus$*-fitting pseudo-multiplication* $\odot : [0,a]^2 \to [0,a]$ as a binary operation with annihilator 0 and neutral element 1 which is non-decreasing and left distributive with respect to $\oplus$, i.e., $(b \oplus c) \odot d = (b \odot d) \oplus (c \odot d)$.

For a fixed pseudo-addition $\oplus$, a fixed $\oplus$-fitting pseudo-multiplication $\odot$, and a capacity $m : 2^X \to [0,1]$, the *discrete Benvenuti integral* $\mathbf{B}_{\oplus,\odot}(m,\cdot) : [0,1]^n \to [0,1]$ is given by

$$\mathbf{B}_{\oplus,\odot}(m,\mathbf{x}) = \bigoplus_{i=1}^{n} (x_{\pi_i} \ominus x_{\pi_{i-1}}) \odot m(\{\pi_i,\ldots,\pi_n\}),$$

where, for $x, y \in [0,a]$ with $x \geqslant y$, the pseudo-difference $x \ominus y$ is given by

$$x \ominus y = \bigvee_{z \in [0,a], y \oplus z = x} z = \sup\{z \in [0,a] | y \oplus z = x\}.$$

Observe that, for the pair $(+,\cdot)$ (i.e., standard addition and multiplication on $[0,\infty]$), $\mathbf{B}_{+,\cdot}$ is just the Choquet integral **Ch**, while $\mathbf{B}_{\vee,\wedge}$ coincides with the Sugeno integral **Su**.

3.3. Smallest discrete universal integrals

Another type of universal integral based on a *semicopula* $\odot$ (i.e., a pseudo-multiplication $\odot$ on $[0,1]$ with neutral element 1, see [5]) is the smallest universal integral linked to $\odot$. For a given capacity $m : 2^X \to [0,1]$, $\mathbf{I}_{\odot,m} : [0,1]^n \to [0,1]$ is given by

$$\mathbf{I}_{\odot,m}(\mathbf{x}) = \bigvee_{i=1}^{n} x_i \odot m(\{j \in X | x_j \geqslant x_i\}) = \bigvee_{i=1}^{n} x_{\pi_i} \odot m(\{\pi_i,\ldots,\pi_n\}).$$

Observe that if $\odot = M$ then $\mathbf{I}_{M,m}(\cdot) = \mathbf{K}_M(m,\cdot)$ is the Sugeno integral with respect to the capacity m. The integral $\mathbf{I}_{\Pi,m}$ is known as the Shilkret integral [24] with respect to m, and for a strict t-norm T, $\mathbf{I}_{T,m}$ was introduced and studied in [29].

On the other hand, $\mathbf{I}_{\odot,m}$ can be seen as Benvenuti integral based on $\vee$ and $\odot$, i.e., $\mathbf{I}_{\odot,m}(\cdot) = \mathbf{B}_{\vee,\odot}(m,\cdot)$, but also as a generalization of the Sugeno integral as suggested in [34]. In general, $\mathbf{I}_{\odot,m}$ is comonotone maxitive and, evidently, idempotent.

4. Axiomatic characterization of some classes of universal integrals

This section is devoted to the axiomatic characterizations of the special classes of discrete universal integrals described in Section 3.

4.1. Axiomatic characterization of discrete copula-based universal integrals

As already mentioned, for each copula $C:[0,1]^2\to[0,1]$ and each capacity $m:2^X\to[0,1]$, the function $\mathbf{K}_C(m,\cdot):[0,1]^n\to[0,1]$ can be seen as an aggregation function. Due to (4), it is not difficult to check that for a constant score vector $\mathbf{c}=(c,\ldots,c)\in[0,1]^n$ we get $\mathbf{K}_C(m,\mathbf{c})=c$, i.e., $\mathbf{K}_C(m,\cdot)$ is an idempotent aggregation function (*unanimous utility function*).

Proposition 4.1. *Let C be a copula and m a capacity on X. Then* $\mathbf{K}_C(m,\cdot)$ *is a comonotone modular aggregation function, i.e., for all comonotone* $\mathbf{x},\mathbf{y}\in[0,1]^n$

$$\mathbf{K}_C(m,\mathbf{x}\vee\mathbf{y})+\mathbf{K}_C(m,\mathbf{x}\wedge\mathbf{y})=\mathbf{K}_C(m,\mathbf{x})+\mathbf{K}_C(m,\mathbf{y}).$$

Proof. Consider two arbitrary comonotone score vectors $\mathbf{x}=(x_1,\ldots,x_n)\in[0,1]^n$ and $\mathbf{y}=(y_1,\ldots,y_n)\in[0,1]^n$, and an arbitrary permutation $(\pi_1,\ldots,\pi_n)$ of $\{1,\ldots,n\}$. Using the notation $\mathbf{x}_{\pi_i}=x_{\pi_i}$ we get $(\mathbf{x}\vee\mathbf{y})_{\pi_i}=\mathbf{x}_{\pi_i}\vee\mathbf{y}_{\pi_i}$ and $(\mathbf{x}\wedge\mathbf{y})_{\pi_i}=\mathbf{x}_{\pi_i}\wedge\mathbf{y}_{\pi_i}$.

Applying (5), we see that

$$\begin{aligned}
&\mathbf{K}_C(m,\mathbf{x}\vee\mathbf{y})+\mathbf{K}_C(m,\mathbf{x}\wedge\mathbf{y})\\
&=\sum_{i=1}^{n}\Big(C((\mathbf{x}\vee\mathbf{y})_{\pi_i},m(\{\pi_i,\ldots,\pi_n\}))+C((\mathbf{x}\wedge\mathbf{y})_{\pi_i},m(\{\pi_i,\ldots,\pi_n\}))\\
&\quad-C((\mathbf{x}\vee\mathbf{y})_{\pi_i},m(\{\pi_{i+1},\ldots,\pi_n\}))-C((\mathbf{x}\wedge\mathbf{y})_{\pi_i},m(\{\pi_{i+1},\ldots,\pi_n\}))\Big)\\
&=\sum_{i=1}^{n}\Big(C(\mathbf{x}_{\pi_i},m(\{\pi_i,\ldots,\pi_n\}))+C(\mathbf{y}_{\pi_i},m(\{\pi_i,\ldots,\pi_n\}))\\
&\quad-C(\mathbf{x}_i,m(\{\pi_{i+1},\ldots,\pi_n\}))-C(\mathbf{y}_i,m(\{\pi_{i+1},\ldots,\pi_n\}))\Big)\\
&=\mathbf{K}_C(m,\mathbf{x})+\mathbf{K}_C(m,\mathbf{y}).\qquad\Box
\end{aligned}$$

Note that neither the idempotency and the modularity nor the comonotone modularity of an aggregation function $U:[0,1]^n\to[0,1]$ imply that U is a copula-based universal integral. For more details about modular aggregation functions see [8,11].

Example 4.2. Define $U:[0,1]^2\to[0,1]$ by $U(x,y)=\left(x\wedge\frac{1}{2}\right)+\left(\left(y-\frac{1}{2}\right)\vee 0\right)$. Then U is an idempotent modular (and thus also comonotone modular) aggregation function with $U(x,y)=x$ for all $(x,y)\in\left[0,\frac{1}{2}\right]^2$. Suppose that $U=\mathbf{K}_C(m,\cdot)$ for some copula C and some capacity m on $X=\{1,2\}$. Then necessarily $m(\{1\})=m(\{2\})=\frac{1}{2}$, i.e., m is a symmetric capacity, and for all $x\leqslant y\leqslant\frac{1}{2}$ we get (compare (4)),

$$\begin{aligned}
x=U(x,y)&=\mathbf{K}_C(m,(x,y))=C(x,1)-C\left(x,\frac{1}{2}\right)+C\left(y,\frac{1}{2}\right)-C(y,0)\\
&=x-C\left(x,\frac{1}{2}\right)+C\left(y,\frac{1}{2}\right).
\end{aligned}$$

Hence $C(x,\frac{1}{2})=C(y,\frac{1}{2})$ also for $x=0$ and $y=\frac{1}{2}$, i.e., $C(\frac{1}{2},\frac{1}{2})=0$. On the other hand, supposing $y\leqslant x\leqslant\frac{1}{2}$,

$$\begin{aligned}
x=U(x,y)&=\mathbf{K}_C(m,(x,y))=C(y,1)-C\left(y,\frac{1}{2}\right)+C\left(x,\frac{1}{2}\right)-C(x,0)\\
&=y-C\left(y,\frac{1}{2}\right)+C\left(x,\frac{1}{2}\right).
\end{aligned}$$

Putting $y=0$ and $x=\frac{1}{2}$, we see that $C(\frac{1}{2},\frac{1}{2})=\frac{1}{2}$, a contradiction.

Proposition 4.3. *Let C be a copula and m a capacity on X. Then*

(i) *for all* $u\in[0,1]$ *and for all* $E,F\subseteq X$ *with* $m(E)=m(F)$

$$\mathbf{K}_C(m,u\cdot\mathbf{1}_E)=\mathbf{K}_C(m,u\cdot\mathbf{1}_F);$$

(ii) *for all* $(u,v)\in[0,1]^2$ *with* $u<v$ *and for all* $E,F\subseteq X$ *with* $m(E)\leqslant m(F)$

$$\mathbf{K}_C(m,u\cdot\mathbf{1}_F)-\mathbf{K}_C(m,u\cdot\mathbf{1}_E)\leqslant\mathbf{K}_C(m,v\cdot\mathbf{1}_F)-\mathbf{K}_C(m,v\cdot\mathbf{1}_E).$$

Proof. Assertion (i) follows from $\mathbf{K}_C(m,u\cdot\mathbf{1}_E)=C(u,m(E))$, and (ii) is a direct consequence of the supermodularity (3) of copulas. $\Box$

Now we are ready to give an axiomatic characterization of discrete copula-based universal integrals.

Theorem 4.4. *Let* $U:[0,1]^n\to[0,1]$ *be an aggregation function. Then the following are equivalent:*

(i) *there is a copula C and a capacity m on X such that* $U(\cdot)=\mathbf{K}_C(m,\cdot)$;

(ii) *U is idempotent and comonotone modular, and for all* $E,F\subseteq X$ *and all* $(u,v)\in[0,1]^2$ *we have*

$$U(\mathbf{1}_E)=U(\mathbf{1}_F)\Rightarrow U(u\cdot\mathbf{1}_E)=U(u\cdot\mathbf{1}_F);\tag{6}$$

$$\begin{aligned}&u\leqslant v\text{ and }U(\mathbf{1}_E)\leqslant U(\mathbf{1}_F)\Rightarrow\\ &U(u\cdot\mathbf{1}_F)-U(u\cdot\mathbf{1}_E)\leqslant U(v\cdot\mathbf{1}_F)-U(v\cdot\mathbf{1}_E).\end{aligned}\tag{7}$$

Proof. From Propositions 4.1 and 4.3 it follows that (i) implies (ii).

Conversely, suppose that (ii) is satisfied, and define the set function $m:2^X\to[0,1]$ by $m(E)=U(\mathbf{1}_E)$. Evidently, m is a capacity on X. Moreover, define the function $D:[0,1]\times\mathrm{Ran}(m)\to[0,1]$ by $D(u,t)=U(u\cdot\mathbf{1}_E)$, where E is some subset of X satisfying $m(E)=t$. Note that D is well-defined because of $U(u\cdot\mathbf{1}_E)=U(u\cdot\mathbf{1}_F)$ whenever $m(E)=m(F)=t$. Moreover, $D(1,t)=U(\mathbf{1}_E)=m(E)=t$ and, due to the idempotency of U, also $D(u,1)=U(u\cdot\mathbf{1}_X)=U(\mathbf{u})=u$. Also, D is supermodular on its domain as a consequence of (7).

Summarizing, $D:[0,1]\times\mathrm{Ran}(m)\to[0,1]$ has all the properties required for a copula, only its domain $[0,1]\times\mathrm{Ran}(m)$ is a proper subset of $[0,1]^2$. Since the domain of D contains the four corner points $(0,0)$, $(0,1)$, $(1,0)$, and $(1,1)$, D is said to be a *subcopula* [21]. From [21, Lemma 2.3.5] we know that for each subcopula D there is a copula C (defined on $[0,1]^2$) which is an extension of D, i.e., it coincides with D on the set $[0,1]\times\mathrm{Ran}(m)$.

The comonotone modularity of U implies for all $\mathbf{x}\in[0,1]^n$

$$U(\mathbf{x}\cdot\mathbf{1}_{\{\pi_n,\pi_{n-1}\}})=U(x_{\pi_n}\cdot\mathbf{1}_{\{\pi_n\}})+U(x_{\pi_{n-1}}\cdot\mathbf{1}_{\{\pi_n,\pi_{n-1}\}})-U(x_{\pi_{n-1}}\cdot\mathbf{1}_{\{\pi_n\}}),$$

where $\mathbf{x}\cdot\mathbf{y}=(x_1\cdot y_1,\ldots,x_n\cdot y_n)$ and $x\cdot\mathbf{y}=(x\cdot y_1,\ldots,x\cdot y_n)$, and, by induction,

$$U(\mathbf{x})=\sum_{i=1}^{n}(U(x_{\pi_i}\cdot\mathbf{1}_{\{\pi_n,\ldots,\pi_i\}})-U(x_{\pi_i}\cdot\mathbf{1}_{\{\pi_n,\ldots,\pi_{i+1}\}})),$$

where the set $\{\pi_{n+1},\pi_n\}$ occurring in the last summand is defined to be the empty set $\emptyset$. However, the last equality means that

$$\begin{aligned}
U(\mathbf{x})&=\sum_{i=1}^{n}(D(x_{\pi_i},m(\{\pi_n,\ldots,\pi_i\})))-D(x_{\pi_i},m(\{\pi_n,\ldots,\pi_{i+1}\}))\\
&=\sum_{i=1}^{n}(C(x_{\pi_i},m(\{\pi_i,\ldots,\pi_n\})))-C(x_{\pi_i},m(\{\pi_{i+1},\ldots,\pi_n\}))\\
&=\mathbf{K}_C(m,\mathbf{x}).\qquad\Box
\end{aligned}$$

Note that, for a given aggregation function U satisfying (ii) in Theorem 4.4, the copula C in (i) is not necessarily unique. This follows from the fact that the extension C of a subcopula D existing because of [21, Lemma 2.3.5] is not necessarily unique (if, e.g., we consider the subcopula $D:\{0,1\}^2\to[0,1]$ given by $D(0,0)=D(0,1)=D(1,0)=0$ and $D(1,1)=1$ then every copula C is an extension of D).

Recall that the symmetry of an aggregation function $U:[0,1]^n \to [0,1]$ means that we have $U(x_1,\ldots,x_n) = U(x_{\pi_1},\ldots,x_{\pi_n})$ for each permutation $(\pi_1,\ldots,\pi_n)$ of $X=\{1,\ldots,n\}$. Similarly, symmetry of a capacity m on X means that we have $m(E)=m(\{\pi_i|i\in E\})$ for each $E\subseteq X$ and for each permutation $(\pi_1,\ldots,\pi_n)$ of X, i.e., $m(E)=m(F)$ whenever $E,F\subseteq X$ have the same cardinality. Surprisingly, symmetry is a sufficient condition to ensure the validity of the properties (6) and (7).

Proposition 4.5. *Let $U:[0,1]^n\to[0,1]$ be an idempotent, comonotone modular aggregation function. If U is symmetric then U satisfies the properties (6) and (7).*

Proof. Fix $(u,v)\in[0,1]^2$ and $E,\ F\subseteq X$ such that $u\leqslant v$ and $U(\mathbf{1}_E)\leqslant U(\mathbf{1}_F)$. Without loss of generality, we can assume $\mathrm{card}(E)\leqslant\mathrm{card}(F)$. Due to the symmetry of U there is $G\subseteq X$ such that $U(\mathbf{1}_G)=U(\mathbf{1}_F)$, $\mathrm{card}(G)=\mathrm{card}(F)$ and $E\subseteq G$. Moreover, $U(t\cdot\mathbf{1}_G)=U(t\cdot\mathbf{1}_F)$ for each $t\in[0,1]$. The comonotone modularity of U ensures $U(u\cdot\mathbf{1}_G)+U(v\cdot\mathbf{1}_E)=U((u\cdot\mathbf{1}_G)\vee(v\cdot\mathbf{1}_E))+U(u\cdot\mathbf{1}_E)$. From the monotonicity of U we have $U(v\cdot\mathbf{1}_G)\geqslant U((u\cdot\mathbf{1}_G)\vee(v\cdot\mathbf{1}_E))$. Thus

$$\begin{aligned}U(u\cdot\mathbf{1}_F)-U(u\cdot\mathbf{1}_E)&=U(u\cdot\mathbf{1}_G)-U(u\cdot\mathbf{1}_E)\\&=U((u\cdot\mathbf{1}_G)\vee(v\cdot\mathbf{1}_E))-U(v\cdot\mathbf{1}_E)\\&\leqslant U(v\cdot\mathbf{1}_G)-U(v\cdot\mathbf{1}_E)=U(v\cdot\mathbf{1}_F)-U(v\cdot\mathbf{1}_E),\end{aligned}$$

i.e., (7) is satisfied. Moreover, if $U(\mathbf{1}_E)=U(\mathbf{1}_F)$ and $v=1$ then

$$0\leqslant U(u\cdot\mathbf{1}_F)-U(u\cdot\mathbf{1}_E)\leqslant U(\mathbf{1}_F)-U(\mathbf{1}_E)=0,$$

i.e., $U(u\cdot\mathbf{1}_E)=U(u\cdot\mathbf{1}_F)$, showing the validity of (6). □

Summarizing, we have proven the following result:

Theorem 4.6. *Let $U:[0,1]^n\to[0,1]$ be a symmetric aggregation function. Then the following are equivalent:*

(i) *there is a copula C and a symmetric capacity m on X such that $U(\cdot)=\mathbf{K}_C(m,\cdot)$;*
(ii) *U is idempotent and comonotone modular.*

Remark 4.7

(i) Recently in [17] the so-called *Ordered Modular Averages* (*OMA operators*) were introduced. These aggregation functions coincide with those characterized in Theorem 4.6.
(ii) In the special case when $C=\Pi$ the classical OWA operators are obtained, i.e., $\mathbf{K}_\Pi(m,\cdot)$ with respect to a symmetric capacity m is just an OWA operator as introduced in [30]. These operators were characterized up to symmetry by the comonotone additivity [7]. For more details about and several applications of OWA operators see [2,6,9,10,28,31–33]. Similarly, $\mathbf{K}_M(m,\cdot)$ with respect to a symmetric capacity m is an *Ordered Weighted Maximum* (*OWMax operator*). It is characterized by symmetry, comonotone maxitivity and $\wedge$-homogeneity [16].

4.2. Axiomatic characterization of the Benvenuti integral

The construction described in [1] yields an axiomatic characterization of the Benvenuti integral in special situations (covering both the Choquet and the Sugeno integral).

Theorem 4.8. *Fix $a\in[1,\infty]$ and let $\oplus:[0,a]^2\to[0,a]$ be a pseudo-addition, $\odot:[0,a]^2\to[0,a]$ be a $\oplus$-fitting pseudo-multiplication which is also associative and right-distributive over $\oplus$, and $U:[0,1]^n\to[0,1]$ be an aggregation function. Then the following are equivalent:*

(i) *there is a capacity $m:2^X\to[0,1]$ so that $U(\cdot)=\mathbf{B}_{\oplus,\odot}(m,\cdot)$;*
(ii) *U is comonotone $\oplus$-additive, i.e., for all comonotone $\mathbf{x}$, $\mathbf{y}\in[0,1]^n$ with $\mathbf{x}\oplus\mathbf{y}\in[0,1]^n$ we have $U(\mathbf{x}\oplus\mathbf{y})=U(\mathbf{x})\oplus U(\mathbf{y})$, and $\odot$-homogeneous, i.e., for all $t\in[0,1]$ and $\mathbf{x}\in[0,1]^n$ we have $U(t\odot\mathbf{x})=t\odot U(\mathbf{x})$.*

Observe that the Benvenuti integral on abstract spaces was defined axiomatically in [1], considering the comonotone $\oplus$-additivity as one of axioms. However, the Benvenuti integral (as introduced in [1]) on a discrete universe $X=\{1,2,\ldots,n\}$ restricted to $[0,1]^n$ is neither an aggregation function nor a $\odot$-homogeneous functional, in general. Thus the result we gave in Theorem 4.8 is an alternative axiomatic approach to a special class of $\odot$-homegeneous discrete Benvenuti integrals in the sense of Section 3.2.

A special class of Benvenuti integrals based on $\oplus=\vee$ is axiomatically characterized in the following subsection.

4.3. Axiomatic characterization of the smallest universal integrals

Based on recent results in [18] we have the following axiomatic characterization of the smallest universal integrals.

Theorem 4.9. *Let $U:[0,1]^n\to[0,1]$ be an idempotent aggregation function. Then the following are equivalent:*

(i) *there is a capacity $m:2^X\to[0,1]$ and a semicopula $\odot:[0,1]^2\to[0,1]$ such that $U=\mathbf{I}_{\odot,m}$;*
(ii) *U is comonotone maxitive and for all $E,\ F\subseteq X$ with $U(\mathbf{1}_E)\leqslant U(\mathbf{1}_F)$ and for each $t\in]0,1[$ we have $U(t\cdot\mathbf{1}_E)\leqslant U(t\cdot\mathbf{1}_F)$.*

Proof. In order to show that (i) implies (ii), the comonotone maxitivity of U was mentioned already in Section 3.3. The monotonicity of $\odot$ ensures

$$U(t\cdot\mathbf{1}_E)=t\odot m(E)\leqslant t\odot m(F)=U(t\cdot\mathbf{1}_F)$$

because of $m(E)=U(\mathbf{1}_E)\leqslant U(\mathbf{1}_E)=m(F)$.

Conversely, assume that (ii) holds, i.e., U is comonotone maxitive, and define $m:2^X\to[0,1]$ by $m(E)=U(\mathbf{1}_E)$. Due to [18, Theorem 1],

$$U(x_1,\ldots,x_n)=\bigvee_{t\in[0,1]}m_t(\{i\in X|x_i\geqslant t\}),$$

where $(m_t)_{t\in[0,1]}$ with $m_t:2^X\to[0,1]$ is a non-decreasing system of monotone set functions such that $m_1(\emptyset)=0$ and $m_1(X)=1$. Consider first the function $\otimes:[0,1]\times\mathrm{Ran}(m)\to[0,1]$ given by $t\otimes m(E)=m_t(E)$ with $E\subseteq X$ which is well-defined because of (ii) and $U(t\cdot\mathbf{1}_E)=m_t(E)$. Now it suffices to define, for instance, $\odot:[0,1]^2\to[0,1]$ by $a\odot b=\lambda\cdot(a\otimes c)+(1-\lambda)\cdot(a\otimes d)$, where $\lambda\in[0,1]$, $c,d\in\mathrm{Ran}(m)$, $c<d$, $]c,d[\cap\mathrm{Ran}(m)=\emptyset$, and $b=\lambda\cdot c+(1-\lambda)\cdot d$ (note that $\odot$ is also well-defined: it is a linear interpolation extending $\otimes$). Evidently, $\odot$ is a semicopula and

$$\begin{aligned}U(x_1,\ldots,x_n)&=\bigvee_{t\in[0,1]}m_t(\{i\in X|x_i\geqslant t\})=\bigvee_{i=1}^n m_{x_{\pi_i}}(\{\pi_i,\ldots,\pi_n\})\\&=\bigvee_{i=1}^n x_{\pi_i}\otimes m(\{\pi_i,\ldots,\pi_n\})=\bigvee_{i=1}^n x_{\pi_i}\odot m(\{\pi_i,\ldots,\pi_n\})\\&=\mathbf{I}_{\odot,m}(x_1,\ldots,x_n).\qquad\Box\end{aligned}$$

When also the symmetry is required we obtain the following axiomatization:

Theorem 4.10. *Let $U:[0,1]^n \to [0,1]$ be a symmetric aggregation function. Then the following are equivalent:*

(i) *There is a symmetric capacity $m:2^X \to [0,1]$ and a semicopula $\odot:[0,1]^2 \to [0,1]$ such that $U = \mathbf{I}_{\odot,m}$;*
(ii) *U is comonotone maxitive and for all $E, F \subseteq X$ with $U(\mathbf{1}_E) = U(\mathbf{1}_F)$ and for each $t \in]0,1[$ we have $U(t \cdot \mathbf{1}_E) = U(t \cdot \mathbf{1}_F)$.*

Proof. It suffices to show that $U(\mathbf{1}_E) < U(\mathbf{1}_F)$ implies $U(t \cdot \mathbf{1}_E) \leqslant U(t \cdot \mathbf{1}_F)$, and then to apply Theorem 4.9. Because of the symmetry of U, $U(\mathbf{1}_E) < U(\mathbf{1}_F)$ implies $\mathrm{card}(E) < \mathrm{card}(F)$ and thus there is a $G \subset F$ such that $\mathrm{card}(G) = \mathrm{card}(E)$. Then $U(t \cdot \mathbf{1}_E) = U(t \cdot \mathbf{1}_G) \leqslant U(t \cdot \mathbf{1}_F)$. □

Example 4.11. Note that we cannot omit the condition that $U(\mathbf{1}_E) = U(\mathbf{1}_F)$ implies $U(t \cdot \mathbf{1}_E) = U(t \cdot \mathbf{1}_F)$ (compare Theorem 4.6). Define $U:[0,1]^2 \to [0,1]$ by $U(x,y) = \wedge(x,y) \vee (\vee(x,y))^2$. Then U is a symmetric, idempotent and comonotone maxitive aggregation function. If we define $m:2^X \to [0,1]$ by $m(E) = U(\mathbf{1}_E)$ we see that $m(E) = 1$ whenever $E \neq \emptyset$. However, then for each semicopula $\odot:[0,1]^2 \to [0,1]$ we get $\mathbf{I}_{\odot,m} = \vee \neq U$.

5. Concluding remarks

We have discussed some classes of discrete universal integrals (which can be seen as special utility functions), focusing on their axiomatic characterization. In the special case when also symmetry is required, this axiomatic characterization is rather simple (for example, idempotency and comonotone modularity in the case of copula-based universal integrals). In future investigations, the results presented here which are valid in the case of a finite universe $X = \{1,\ldots,n\}$ should be discussed for arbitrary abstract measurable spaces $(X,\mathscr{A})$. Here, some additional requirements concerning continuity are to be expected.

Acknowledgments

The authors gratefully acknowledge the support of the bilateral Austrian-Slovakian project SK-AT-0020-08. The second author was supported by the Grants P 402/11/0378, APVV-0073-10 and VEGA 1/0080/10. We are grateful to Andrea Mesiarová-Zemánková for her valuable comments and suggestions during the preparation of this manuscript.

References

[1] P. Benvenuti, R. Mesiar, D. Vivona, Monotone set functions-based integrals, in: E. Pap (Ed.), Handbook of Measure Theory, chapter 33, Elsevier Science, Amsterdam, 2002, pp. 1329–1379.
[2] F.J. Cabrerizo, I.J. Pérez, E. Herrera-Viedma, Managing the consensus in group decision making in an unbalanced fuzzy linguistic context with incomplete information, Knowl.-Based Syst. 23 (2010) 169–181.
[3] G. Choquet, Theory of capacities, Ann. Inst. Fourier (Grenoble) 5 (1953–1954) 131–292.
[4] C. Dellacherie, Quelques commentaires sur les prolongements de capacités, in: Séminaire de Probabilités, V (Univ. Strasbourg, année universitaire 1969–1970), Lecture Notes in Mathematics, vol. 191, Springer, Berlin, 1971, pp. 77–81.
[5] F. Durante, C. Sempi, Semicopulæ, Kybernetika (Prague) 41 (2005) 315–328.
[6] A. Emrouznejad, MP-OWA: the most preferred OWA operator, Knowl.-Based Syst. 21 (2008) 847–851.
[7] M. Grabisch, Fuzzy integral in multicriteria decision making, Fuzzy Sets Syst. 69 (1995) 279–298.
[8] M. Grabisch, J.-L. Marichal, R. Mesiar, E. Pap, Aggregation Functions, Cambridge University Press, Cambridge, 2009.
[9] E. Hermans, D. Ruan, T. Brijs, G. Wets, K. Vanhoof, Road safety risk evaluation by means of ordered weighted averaging operators and expert knowledge, Knowl.-Based Syst. 23 (2010) 48–52.
[10] A.H. Keyhanipour, B. Moshiri, M. Kazemian, M. Piroozmand, C. Lucas, Aggregation of web search engines based on users' preferences in WebFusion, Knowl.-Based Syst. 20 (2007) 321–328.
[11] E.P. Klement, R. Mesiar, Integral-based modifications of OWA-operators, in: S. Li, X. Wang, Y. Okazaki, J. Kawabe, T. Murofushi, L. Guan (Eds.), Nonlinear Mathematics for Uncertainty and its Applications, Springer, Berlin, 2011, pp. 325–331.
[12] E.P. Klement, R. Mesiar, E. Pap, Triangular Norms, Kluwer Academic Publishers, Dordrecht, 2000.
[13] E.P. Klement, R. Mesiar, E. Pap, Measure-based aggregation operators, Fuzzy Sets Syst. 142 (2004) 3–14.
[14] E.P. Klement, R. Mesiar, E. Pap, A universal integral as common frame for Choquet and Sugeno integral, IEEE Trans. Fuzzy Syst. 18 (2010) 178–187.
[15] J. Li, R. Mesiar, P. Struk, Pseudo-optimal measures, Inform. Sci. 180 (2010) 4015–4021.
[16] J.-L. Marichal, An axiomatic approach of the discrete Sugeno integral as a tool to aggregate interacting criteria in a qualitative framework, IEEE Trans. Fuzzy Syst. 9 (2001) 164–172.
[17] R. Mesiar, A. Mesiarová-Zemánková, The ordered modular averages, IEEE Trans. Fuzzy Syst. 19 (2011) 42–50.
[18] R. Mesiar, A. Mesiarová-Zemánková, K. Ahmad, Level-dependent Sugeno integral, IEEE Trans. Fuzzy Syst. 17 (2009) 167–172.
[19] P.S. Mostert, A.L. Shields, On the structure of semi-groups on a compact manifold with boundary, Ann. Math. II. Ser. 65 (1957) 117–143.
[20] T. Murofushi, M. Sugeno, Fuzzy t-conorm integrals with respect to fuzzy measures: generalization of Sugeno integral and Choquet integral, Fuzzy Sets Syst. 42 (1991) 57–71.
[21] R.B. Nelsen, An Introduction to Copulas, Lecture Notes in Statistics, second ed., vol. 139, Springer, New York, 2006.
[22] E. Pap, An integral generated by a decomposable measure, Univ. u Novom Sadu Zb. Rad. Prirod.-Mat. Fak. Ser. Mat. 20 (1) (1990) 135–144.
[23] D. Schmeidler, Integral representation without additivity, Proc. Amer. Math. Soc. 97 (1986) 255–261.
[24] N. Shilkret, Maxitive measure and integration, Indag. Math. 33 (1971) 109–116.
[25] A. Sklar, Fonctions de répartition à n dimensions et leurs marges, Publ. Inst. Statist. Univ. Paris 8 (1959) 229–231.
[26] M. Sugeno, Theory of fuzzy integrals and its applications. Ph.D. thesis, Tokyo Institute of Technology, 1974.
[27] M. Sugeno, T. Murofushi, Pseudo-additive measures and integrals, J. Math. Anal. Appl. 122 (1987) 197–222.
[28] J. Vaníček, I. Vrana, S. Aly, Fuzzy aggregation and averaging for group decision making: a generalization and survey, Knowl.-Based Syst. 22 (2009) 79–84.
[29] S. Weber, Two integrals and some modified versions: critical remarks, Fuzzy Sets Syst. 20 (1986) 97–105.
[30] R.R. Yager, On ordered weighted averaging aggregation operators in multicriteria decision making, IEEE Trans. Syst. Man Cybernet. 18 (1988) 183–190.
[31] R.R. Yager, G. Gumrah, M.Z. Reformat, Using a web Personal Evaluation Tool–PET for lexicographic multi-criteria service selection, Knowl.-Based Syst. 24 (2011) 929–942.
[32] R.R. Yager, J. Kacprzyk (Eds.), The Ordered Weighted Averaging Operators, Theory and Applications, Springer, Berlin, 1997.
[33] R.R. Yager, J. Kacprzyk, G. Beliakov (Eds.), Recent Developments in the Ordered Weighted Averaging Operators: Theory and Practice, Springer, Berlin, 2011.
[34] R. Zhao, (N)-Fuzzy integral, J. Math. Res. Expo. 2 (1981) 55–72. In Chinese.